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1991 J. Phys. A: Math. Gen. 24 L1129

(http://iopscience.iop.org/0305-4470/24/19/001)

J. Phys. A: Math. Gen. 24 (1991) L1129–L1132. Printed in the UK

LETTER TO THE EDITOR

Nonlinear shift of q-Bose operators and q-coherent states

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Received 26 June 1991

Abstract. The analogue of shift for q-Bose operators is found. In contrast with the case of ordinary Bose operators this shift is a nonlinear transformation: unitary operators of shift do not form a group. Two dual systems of q-coherent states associated with these shift operators are obtained. The matrix elements of the shift operator in canonical basis are expressed via the Charlier q-polynomials.

The so-called q-oscillator probably plays the role of an ordinary oscillator at very small distances of the order of Planck length (Biedenharn 1989).

To construct the q-oscillator one needs to define the q-Bose creation A_+ and annihilation A_- operators by the following relation (Biedenharn 1989, Kulish and Damaskinsky 1990)

$$A_-A_+ - qA_+A_- = 1 \tag{1}$$

where $q = \exp(-\omega)$, $\omega \geqslant 0$.

The relation (1) is the q-analogue of the usual commutation rule defining the Heisenberg algebra of the ordinary boson creation a^+ and annihilation a operators. If $\omega \to 0$ then $A_+ \to a^+$, $A_- \to a$. Sometimes the algebra of operators A_+, A_- is called the q-Heisenberg algebra.

One can also introduce the number operator A_0 by the relation

$$\exp(-\omega A_0) = [A_-, A_+]. \tag{2}$$

Then three operators A_+, A_-, A_0 form the so-called q-oscillator algebra. There exists the canonical basis $|n\rangle$ for this algebra:

$$\begin{aligned} &A_0|n\rangle = n|n\rangle \\ &A_-|n\rangle = \mu_n|n-1\rangle \qquad n = 0, 1, \ldots \\ &A_+|n\rangle = \mu_{n+1}|n+1\rangle \end{aligned} \tag{3}$$

where $\mu_n = (1-q^n)/(1-q)$.

The q-coherent states (q-CS) $|z\rangle_q$ are defined to be eigenstates for the annihilation operator A_- (Biedenharn 1989, Kulish and Damaskinsky 1990, Quesne 1991)

$$A_-|z\rangle_q = z|z\rangle_q. \tag{4}$$

If $q \to 1$ then the q-CS $|z\rangle_q$ turn into the ordinary (Glauber) CS $|z\rangle$.

The q-CS possess some remarkable properties analogous to those of the ordinary CS. For example, there is the following relation

$$|z\rangle_q = e_q(zA_+)|0\rangle/\sqrt{e_q(|z|^2)} \tag{5}$$

0305-4470/91/191129+04$03.50

where

$$e_q(z) = \sum_{n=0}^{\infty} z^n/[n!]_q$$

is the q-exponent and

$$[n!]_q = (1-q)(1-q^2)\dots(1-q^n)/(1-q)^n.$$

However for the ordinary CS there is another (but equivalent) method of definition. This is based on the existence of a shift automorphism for the Heisenberg algebra:

$$b(z) = D(z)aD^+(z) = a - z \tag{6}$$

where

$$D(z) = \exp(za^+ - \bar{z}a) \tag{7}$$

is a unitary operator shifting the Bose operators by the parameter z (Perelomov 1986).

For the real values of z the operators $D(z)$ form a one-parameter subgroup and z is an additive parameter:

$$\begin{aligned} &D(x)D(y) = D(x+y) \\ &D^+(z) = D(-z). \end{aligned} \tag{8}$$

The ordinary CS can be defined as a vacuum for the shifted annihilation operator b:

$$|z\rangle = D(z)|0\rangle \qquad b(z)|z\rangle = 0. \tag{9}$$

One can ask about the existence of a shift transformation for the q-Bose operators. It is clear that a simple shift like (6) is not an automorphism, i.e. it does not conserve the relation (1). So such a transformation, if it exists, should be nonlinear.

One can easily verify that this nonlinear shift really exists and has the form

$$\begin{aligned} &B_-(z) = A_-\sqrt{1 - v\exp(-\omega A_0)} - z\exp(-\omega A_0) \\ &B_+(z) = \sqrt{1 - v\exp(-\omega A_0)}A_+ - z\exp(-\omega A_0) \end{aligned} \tag{10}$$

where $v = z^2(q^{-1} - 1)$, the parameter z is taken to be real and

$$z^2 < 1/(q^{-1} - 1).$$

The transformation (10) is an automorphism of the algebra (1), i.e.

$$B_-(z)B_+(z) - qB_+(z)B_-(z) = 1. \tag{11}$$

Therefore there should exist a unitary operator $U(z)$ being the q-analogue of $D(z)$:

$$U(z)A_\pm U^+(z) = B_\pm(z). \tag{12}$$

One can define the new system of q-coherent states $\widetilde{|z\rangle}_q$ coinciding with the vacuum of the $B_0(z)$:

$$\widetilde{|z\rangle}_q = U(z)|0\rangle \qquad B_-\widetilde{|z\rangle}_q = 0. \tag{13}$$

In contrast with the case of ordinary CS the systems $|z\rangle_q$ and $\widetilde{|z\rangle}_q$ do not coincide:

$$\widetilde{|z\rangle}_q \neq \widetilde{|z\rangle}_q \qquad \text{for } z \neq 0. \tag{14}$$

The reason for this is that the operators $U(z)$ do not form *a* group, i.e.

$$U(x)U(y) \neq U(z) \qquad U^{+}(z) \neq U(-z) \tag{15}$$

(relation (15) follows from the fact that two consequent nonlinear shifts (10) are not the third shift).

Nevertheless, there exists a simple relation between $|z\rangle_q$ and $\widetilde{|z\rangle}_q$. To see this relation let us operate by the $B_-(-z)$ to the vacuum $|0\rangle$:

$$B_-(-z)|0\rangle = z|0\rangle. \tag{16}$$

So the vacuum $|0\rangle$ is a q-coherent state (in the sense of (4)) for the operator $B_-(-z)$. Therefore

$$|z\rangle_q = U^{+}(-z)|0\rangle. \tag{17}$$

The relations (13) and (17) define two different systems of q-coherent states. One can say that the system $\widetilde{|z\rangle}_q$ is dual with regard to $|z\rangle_q$. There is the following relation between these systems:

$$\widetilde{|z\rangle}_q = U(z)U(-z)|z\rangle_q. \tag{18}$$

Let us examine the structure of matrix coefficients $M_{mn} = \langle m|U^{+}(z)|n\rangle$ for the operator $U^{+}(z)$.

On the one hand

$$\langle m|\exp(-\omega A_0)U^{+}(z)|n\rangle = \exp(-\omega m)M_{mn}. \tag{19}$$

On the other hand

$$\begin{aligned}\langle m|\exp(-\omega A_0)U^{+}(z)|n\rangle &= \langle m|U^{+}(z)U(z)\exp(-\omega A_0)U^{+}(z)|n\rangle \\ &= \langle m|U^{+}(z)[B_-(z), B_+(z)]|n\rangle \\ &= d_{n+1}M_{mn+1} + d_nM_{mn-1} + g_nM_{mn}\end{aligned} \tag{20}$$

where

$$\begin{aligned} g_n &= (1+v)q^n - (1+q)vq^{2n} \\ d_n &= zq^{n-1}[(1-q)(1-q^n)(1-vq^n)]^{1/2}. \end{aligned} \tag{21}$$

It is seen from (20) and (19) that one can represent the matrix elements $M_{mn}(z)$ in the form:

$$M_{mn}(z) = \langle m|U^{+}(z)|0\rangle C_n(x(m)) \tag{22}$$

where $C_n(x(m))$ is the system of orthogonal polynomials of the discrete argument $x(m) = \exp(-\omega m)$:

$$\sum_{m=0}^{\infty} W_m C_n(x(m))C_{n'}(x(m)) = \delta_{nn'}. \tag{23}$$

The weight function W_m for these polynomials has the form

$$W_m = (\langle m|U^{+}(z)|0\rangle)^2 = (\langle m|-z\rangle_q)^2 = z^{2m}/(e_q(z^2)[m!]_q). \tag{24}$$

One can conclude that the weight function (24) coincides with that for the so-called Charlier q-polynomials (we define them as in Nikiforov *et al* (1985)), which are the special case of the Askey–Wilson q-polynomials (Askey and Wilson 1985).

If $\omega \to 0$ the matrix element $M_{mn}(z)$ can be expressed in terms of ordinary Charlier polynomials (Perelomov 1986, Feinsilver 1988). So the formula (22) is the q-generalization of the 'classical' result.

The remaining problems are:

(i) to obtain the explicit form of the 'shift' operator $U(z)$ in terms of q-exponents;

(ii) to find the nonlinear automorphisms for other q-algebras. For example it would be interesting to find the nonlinear 'rotation' in $SU_q(2)$.

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